Singular Locally-Scalar Representation of Quivers in Hilbert Spaces and Separating Functions

I.K. Redchuk $*$, A.V. Roiter **

* Institute of Mathematics of National Academy of Sciences of Ukraine,
Tereshchenkovska str., 3, Kiev, Ukraine, ind. 01601
E-mail: red@imath.kiev.ua

** Institute of Mathematics of National Academy of Sciences of Ukraine,
Tereshchenkovska str., 3, Kiev, Ukraine, ind. 01601
E-mail: roiter@imath.kiev.ua

In the classical works [1], [2] representations of quivers (in the category of finite-dimensional vector spaces) were considered and their connection with root systems of corresponding graphs was established, which developing further in the works [3]–[6]. Repeatedly attempts were made to generalize representations of quivers to metric and, in particular, Hilbert spaces, but at that mentioned connection was lost. In [7] a restriction of *local scalarity* was imposed on representations of graphs in Hilbert spaces, and after that it was managed to generalize the results of [1], [2] in a natural way by constructing, analogous to Coxeter functors, functors of even and add reflections, moreover it turned out that just these representations are of interest of functional analysis and in some particular cases in other terms in fact were considered in [8]–[9].

In this paper we will consider a connection of locally-scalar representations of extended Dynkin graphs with function ρ considered in [10], and also functions ρ_k playing analogous part for more wide class of graphs.

1 Standard singular representations of extended Dynkin graphs.

We will widely use denotations and definitions of the article [7].

All considered graphs will be supposed finite, connected and acyclic (i. e. woods).

Multiplicity $\mu(g)$ of a vertex $g \in G_v$ is $|M_g|$ ($M_g = \{g_i \in G_v \,|\, g_i - g\}$ [7]).

Hypothesis 1. *All indecomposable locally-scalar representation of a graph G are finite-dimensional, if and only if G is a Dynkin graph or extended Dynkin graph.*

In this paper we will consider only finite-dimensional representations.

Let us fix a separation of set of the vertices G_v of graph G to even $\overset{\circ}{G}_v$ and odd ones $\overset{\bullet}{G}_v$ [7] and a numeration $g_1, \dots, g_n$, numbering (in an arbitrary order) at first odd $g_1, g_2, \dots, g_p$ and then even vertices $g_{p+1}, g_{p+2}, \dots, g_n$. Let $x \in V_G$, ($x : G_v \to \mathbb{C}$) $x_i = x(g_i)$, c — Coxeter

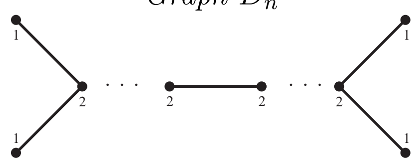
transformation on V_G, $c = \sigma_{g_n}\sigma_{g_{n-1}}\cdots\sigma_{g_1}$, $(\sigma_{g_i}(x))_i = -x_i + \sum\limits_{j,\, g_j \in M_{g_i}} x_j$, $(\sigma_{g_i}(x))_j = x_j$ when $j \neq i$. It is clear that $\sigma_i^2 = \mathrm{id}$ $(i = \overline{1,n})$. Therefore $c^{-1} = \sigma_{g_1}\cdots\sigma_{g_n}$.

Vector $x \in V_G^+$ is *regular* if $c^t(x) \in V_G^+$ for any $t \in \mathbb{Z}$ and *singular* in a contrary case (terminology traces back to [12]).

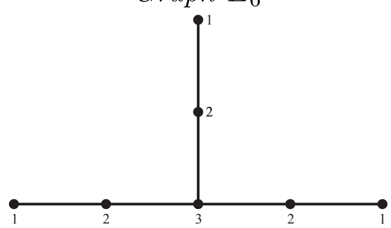
The main part in our study will play extended Dynkin graphs ($\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$). Their connection with Coxeter transformations can be characterized by the following well-known statement.

Lemma 1.1. *If G is a wood, $u_G = u \in V_G^+$ and $c(u) = u$, then G is an extended Dynkin graph and (up to the common multiplier) u_G looks like:*

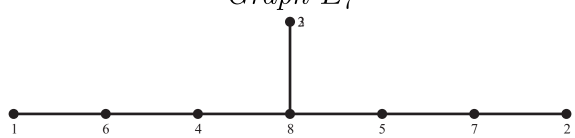
Graph $\widetilde{D}_n$



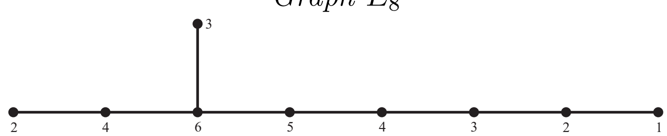
Graph $\widetilde{E}_6$



Graph $\widetilde{E}_7$



Graph $\widetilde{E}_8$



Proof.

$c(u) = u$ implies $\sigma_{g_i}(u) = u$ (and this is equivalent to the statement, that when $i = \overline{1,n}$ $2u(g_i) = \sum\limits_{g \in M_{g_i}} u(g)$) and $u_i > 0$ when $i \in \overline{1,n}$. Vertex g is the *point of branching* if $\mu(g) > 2$; g — *point of weak branching* if $\mu(g) = 3$ and $|M_g| = \{a,b,c\}$ where $\mu(a) = \mu(b) = 1$. $\sigma_{g_i} = u$ implies:

1) if $a - b$ then $u(a) \geq \frac{1}{2}u(b)$, at that $u(a) = \frac{1}{2}u(b)$ only if $\mu(a) = 1$;

2) if $\mu(g) > 2$, $g - a$ then $u(a) \leq u(g)$, at that $u(g) = u(a)$ only if $M_g = \{a,b,c\}$ and $\mu(b) = \mu(c) = 1$ (it follows from 1));

3) if $a - b - c$ and $u(b) \leq (a)$ then $u(c) \leq u(b)$, and if $u(b) < u(a)$ then $u(c) < u(b)$.

If G_v contains 2 points of branching x and y, $x - z_1 - \cdots - z_t - y$ $(t \geq 0)$ and $z_1, \ldots, z_t$ are not points of branching then $u(x) \geq u(z_1) \geq \ldots u(z_t) \geq u(y)$ and $u(y) \geq u(z_1) \geq \ldots u(z_t) \geq u(x)$. Thus, 2) implies x and y are points of weak branching and $G = \widetilde{D}_n$.

If there is only one point of branching z in G then it is easy to see that only the cases $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ and listed above u_G are possible, and the case of missing of the points of branching is impossible. $\Box$

Note that the vector u_G for extended Dynkin graph G is its (unique up to common multiplier) imaginary root. Clearly, u is regular.

It is well-known that if G is a Dynkin graph (A_n, D_n, E_6, E_7, E_8) then all vectors from V_G are singular [2]. Converse is also true: for all other woods there exist also regular vectors. Indeed, if G — extended Dynkin graph then u_G is regular. From the root theory of the arbitrary graph [11] it is follows that any imaginary root is regular (we will obtain it also as a corollary from the proposition 1.5). However, also real roots can be regular: for instance, if $G = \widetilde{E}_6$, $v \in V_G^+$, $v_i = 1$ for $i = \overline{1,7}$.

Locally-scalar representation π of graph a G is *singular* if π is indecomposable and finite-dimensional, and $d(\pi)$ is a singular vector; *regular* if π is indecomposable, finite-dimensional and not singular.

Hypothesis 2. *If G is an extended Dynkin graph, π — its regular representation then $d(\pi) \leq u_G$.*

In [7] for (arbitrary) graph G were considered: a category $\mathrm{Rep}\,(G, \mathcal{H})$ of representations of G in the category of Hilbert spaces, a category $\mathrm{Rep}\,(G)$ of locally-scalar representations, its (full) subcategory $\mathrm{Rep}\,(G, d, f)$ of representations with fixed dimension d and character f, and, finally, $\mathrm{Rep}\,(G, \coprod)$ — a union of the categories $\mathrm{Rep}\,(G, d, f)$ satisfying the condition $d_i + f_i > 0$, $i \in \overline{1,n}$.

Note that all listed categories, except $\mathrm{Rep}(G, \mathcal{H})$, are not additive.

If vector f is such that $f_i > 0$ when $i = \overline{p+1,n}$ then in [7] it was constructed a functor $\overset{\circ}{F}_{df}\colon \mathrm{Rep}\,(G, d, f) \to \mathrm{Rep}\,(G, d^+, \overset{\circ}{f}_d)$ which is equivalence ($d^+ = \overset{\circ}{c}\,(d)$). At that $d_i^+ = d_i$ when $g_i \in \overset{\bullet}{G}$, $d_i^+ = \sigma_i(d)$ when $g_i \in \overset{\circ}{G}$; $(\overset{\circ}{f}_d)_i = f_i$ when $g_i \in \overset{\circ}{G}$, and if $g_i \in \overset{\bullet}{G}$ then $(\overset{\circ}{f}_d)_i = \sigma_i(f)$ when $d_i \neq 0$ and $(\overset{\circ}{f}_d)_i = f_i$ when $d_i = 0$. Analogously functors $\overset{\bullet}{F}_{df}\colon \mathrm{Rep}\,(G, d, f) \to \mathrm{Rep}\,(G, d^-, \overset{\bullet}{f}_d)$ are constructed ($d^- = \overset{\bullet}{c}\,(d)$).

For $d \in V_G^+$ define $G^d = \{g \in G_v | d(g) > 0,\}$. We will need slightly another construction of functors $\overset{\circ}{\Phi}_{df}\colon \mathrm{Rep}\,(G, d, f) \to \mathrm{Rep}\,(G, d^+, \overset{\circ}{f'}_d)$ on condition that $f(g_i) > 0$ for $g_i \in (G^d \bigcup M(G^d)) \bigcap \overset{\circ}{G}$. These functors, as functors $\overset{\circ}{F}_{df}$, results from functors $\overset{\circ}{F}_{X,\delta}$ by the selection of other δ (corresponding to $\overset{\circ}{f'}_d$). Here $(\overset{\circ}{f'}_d)_i = f_i$ when $g_i \in \overset{\circ}{G}$, and if $g_i \in \overset{\bullet}{G}$ then when $d_i = 0$, $g_i \in M(G^d)$ or $d_i^+ = 0$, $g_i \in M(G^{d^+})$ we define $(\overset{\circ}{f'}_d)_i = f_i$, and in other cases $(\overset{\circ}{f'}_d)_i = \sigma_i(f)$.

Functoriality of $\overset{\circ}{\Phi}_{df}$ folows from the reasonings analogous to mentioned in [7] during the proof of functoriality of $\overset{\circ}{F}_{df}$ from $\mathrm{Rep}\,(G, d, f)$ to $\mathrm{Rep}\,(G, d^+, \overset{\circ}{f}_d)$).

Analogously, of course, functor $\overset{\bullet}{\Phi}_{df}$ is constructed.

Let $S' = \{(d, f) \in Z_G \times V_G \,|\, f(g) > 0$ when $g \in M(G^d)\}$, $\overset{\circ}{S'} = \{(d, f) \in S' \,|\, f(g) > 0$ when $g \in G^d \bigcap \overset{\circ}{G}\}$, $\overset{\bullet}{S'} = \{(d, f) \in S' \,|\, f(g) > 0$ when $g \in G^d \bigcap \overset{\bullet}{G}\}$.

Let us construct a category $\mathrm{Rep}(G, \coprod') = \coprod\limits_{(d,f)\in S'} \mathrm{Rep}(G, d, f)$. So, objects of $\mathrm{Rep}(G, \coprod')$ are pairs (π, f) where π is a locally-scalar representation, f is its character (which is completely determined by the representation π only if π is faithful; dimension d, of course, is always determined by the representation π). Morphisms between objects from $\mathrm{Ob}\,\mathrm{Rep}(G, d_1, f_1)$ and $\mathrm{Ob}\,\mathrm{Rep}(G, d_2, f_2)$ match with morphisms in $\mathrm{Ob}\,\mathrm{Rep}(G, d_1, f_1)$ when $(d_1, f_1) = (d_2, f_2)$ and are missing on the pairs of object such that $(d_1, f_1) \neq (d_2, f_2)$.

A category $\mathrm{Rep}_\circ(G, \coprod') \subset \mathrm{Rep}(G, \coprod')$ is a full subcategory of f-representations, for which (d, f) such that $f(g) > 0$ when $g \in (M(G^d) \bigcup G^d) \bigcap \overset{\bullet}{G}$ (analogously a category $\mathrm{Rep}_\bullet(G, \coprod')$ is defined).

It is easy to check that if $(d, f) \in \overset{\circ}{S'}$ then $(d^+, \overset{\circ}{f'}) \in \overset{\circ}{S'}$ and if $(d, f) \in \overset{\bullet}{S'}$ then $(d^+, \overset{\bullet}{f'}) \in \overset{\bullet}{S'}$. Thus, a functor $\overset{\circ}{\Phi}$: $\mathrm{Rep}_\circ(G, \coprod') \to \mathrm{Rep}_\circ(G, \coprod')$ — a union of functors $\overset{\circ}{\Phi}_{df}$, and a functor $\overset{\bullet}{\Phi}$: $\mathrm{Rep}_\bullet(G, \coprod') \to \mathrm{Rep}_\bullet(G, \coprod')$ — a union of functors $\overset{\bullet}{\Phi}_{df}$ are determined (analogously $\overset{\circ}{F}$ and $\overset{\bullet}{F}$ in [7]).

$\overset{\circ}{\Phi}{}^2 \cong \mathrm{Id}$ and $\overset{\bullet}{\Phi}{}^2 \cong \mathrm{Id}$. Therefore, $\overset{\circ}{\Phi}$ ($\overset{\bullet}{\Phi}$) is an equivalence in $\mathrm{Rep}_\circ(G, \coprod')$ ($\mathrm{Rep}_\bullet(G, \coprod')$).

Let us introduce denotations: $\overset{\bullet}{c} = \sigma_{g_p} \cdots \sigma_{g_1}$, $\overset{\circ}{c} = \sigma_{g_n} \cdots \sigma_{g_{p+1}}$. Let $c_t = \underbrace{\cdots \overset{\bullet}{c}\overset{\circ}{c}\overset{\bullet}{c}}_{t}$ when $t > 0$, $c_t = \underbrace{\cdots \overset{\circ}{c}\overset{\bullet}{c}\overset{\circ}{c}}_{t}$ when $t < 0$ and $c_0 = \mathrm{id}$.

Operators c_i generate in the group of invertible operators in V_G a subgroup isomorphous to dihedral group; $c_r c_s = c_t$, where $t = (-1)^s r + s$.

Let $\mathrm{Rep}^1(G, \coprod') = \mathrm{Rep}_\circ(G, \coprod') \bigcap \mathrm{Rep}_\bullet(G, \coprod')$, functors $\overset{\circ}{\Phi}$, $\overset{\bullet}{\Phi}$ are defined on $\mathrm{Rep}^1(G, \coprod')$ with values in $\mathrm{Rep}(G, \coprod')$. Let us construct in $\mathrm{Rep}(G, \coprod')$ a sequence of full subcategories

$$\mathrm{Rep}\,(G, \coprod{}') \supset \mathrm{Rep}^1\,(G, \coprod{}') \supset \ldots \supset \mathrm{Rep}^k\,(G, \coprod{}') \supset \ldots$$

such that $\mathrm{Ob}\,\mathrm{Rep}^{i+1}(G, \coprod') = \{X \in \mathrm{Ob}\,\mathrm{Rep}^i(G, \coprod') \mid \overset{\circ}{\Phi} \in \mathrm{Rep}^i(G, \coprod'), \overset{\bullet}{\Phi} \in \mathrm{Rep}^i(G, \coprod')\}$. Thus, on $\mathrm{Rep}^k(G, \coprod')$ functors $\overset{\circ}{\Phi}$, $\overset{\bullet}{\Phi}$ with values in $\mathrm{Rep}^{k-1}(G, \coprod')$ are defined. Hence for each $k \in \mathbb{N}$ functors

$$\Phi_k : \mathrm{Rep}^k\,(G, \coprod{}') \to \mathrm{Rep}\,(G, \coprod{}'),\ \ \Phi_k = \underbrace{\cdots \overset{\bullet}{\Phi}\overset{\circ}{\Phi}\overset{\bullet}{\Phi}}_{k},$$

$$\Phi_{-k} : \mathrm{Rep}^k\,(G, \coprod{}') \to \mathrm{Rep}\,(G, \coprod{}'),\ \ \Phi_{-k} = \underbrace{\cdots \overset{\circ}{\Phi}\overset{\bullet}{\Phi}\overset{\circ}{\Phi}}_{k}$$

are defined.

Formulas listed above imply that if $\Phi_t(\pi, f) = (\pi_t, f_t)$ then $d(\pi_t) = c_t(d(\pi))$.

Let G — extended Dynkin graph, $u_G = (u_k)$. Let us construct a linear form

$$L_G(x) = \sum_{g_i \in \overset{\bullet}{G}} u_i x_i - \sum_{g_j \in \overset{\circ}{G}} u_j x_j,\ \ x \in V_G.$$

Following statement takes place:

Lemma 1.2. *For any $x \in V_G$ $L_G(c_1(x)) = L_G(c_{-1}(x)) = -L_G(x)$.*

Proof.
Denote as $S(x_i) = (\sigma_{g_i}(x))_i + x_i$.

$L_G(c_1(x)) + L_G(x) = \sum\limits_{g_i \in \overset{\bullet}{G}} u_i(\sigma_{g_i}(x))_i - \sum\limits_{g_i \in \overset{\circ}{G}} u_i x_i + \sum\limits_{g_i \in \overset{\bullet}{G}} u_i x_i - \sum\limits_{g_i \in \overset{\circ}{G}} u_i x_i = \sum\limits_{g_i \in \overset{\bullet}{G}} u_i((\sigma_{g_i}(x))_i + x_i) - 2\sum\limits_{g_i \in \overset{\circ}{G}} u_i x_i = \sum\limits_{g_i \in \overset{\bullet}{G}} u_i S(x_i) - 2\sum\limits_{g_i \in \overset{\circ}{G}} u_i x_i = \sum\limits_{g_i \in \overset{\circ}{G}} x_i(S(u_i) - 2u_i) = \sum\limits_{g_i \in \overset{\circ}{G}} x_i((\sigma_{g_i}(u_i))_i - u_i) = 0.$

A case $L_G(c_{-1}(x))$ is examined analogously. □

Let $u = \overset{\bullet}{u} + \overset{\circ}{u}$ where $(\overset{\bullet}{u})_i = u_i$ when $i \leq p$ and $(\overset{\circ}{u})_i = u_i$ when $i > p$. Let then $u_\circ^{|k|} = c_k(\overset{\circ}{u})$ when $k \geq 0$ and $u_\bullet^{|k|} = c_k(\overset{\bullet}{u})$ when $k \leq 0$ and, finally, $\widetilde{u}^k = \alpha u^k$ where $\min\limits_{i=\overline{1,n}} (\widetilde{u}^k)_i = 1$.

Character of the representation π is said to be *odd* (resp. *even*) *standard* if it equals $\widetilde{u}_\bullet^k$ (resp. $\widetilde{u}_\circ^k$). Characters $\widetilde{u}_\circ^0$ and $\widetilde{u}_\bullet^0$ are said to be *primary standard*.

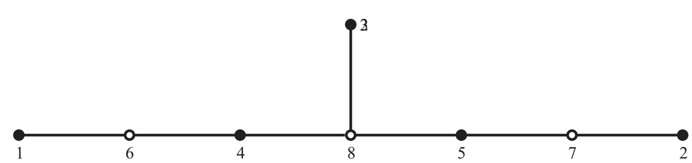
Locally-scalar f-representation of an extended Dynkin graph G is said to be *standard* if f is a standard character.

In [10] (increasing) function ρ: $\rho(n) = 1 + \frac{n-1}{n+1}$, $n \in \mathbb{N}_0$ is defined. (It is naturally to consider $\rho(\infty) = 2$). We will give an explicit formula for the standard characters in terms of ρ and will show for each singular root exactly one standard representation corresponds.

Proposition 1.3. *Let G — extended Dynkin graph. Let $\widetilde{u}_\bullet^0 = (w_1, \ldots, w_p, 0, \ldots, 0)$ and $\widetilde{u}_\circ^0 = (0, \ldots, 0, w_{p+1}, \ldots, w_n)$. Then*
$\widetilde{u}_\bullet^{2m} = (w_1, \ldots w_p, \rho(2m)w_{p+1}, \ldots \rho(2m)w_n)$,
$\widetilde{u}_\bullet^{2m+1} = (w_1, \ldots w_p, (4 - \rho(2m))w_{p+1}, \ldots (4 - \rho(2m))w_n)$,
$\widetilde{u}_\circ^{2m} = (w_1, \ldots w_p, (4 - \rho(2m-1))w_{p+1}, \ldots (4 - \rho(2m-1))w_n)$,
$\widetilde{u}_\circ^{2m+1} = (w_1, \ldots w_p, \rho(2m+1)w_{p+1}, \ldots \rho(2m+1)w_n)$,
$m \in \mathbb{N}_0$.

Proof is carried out separately for each extended Dynkin graph. We will illustrate the proof by the example of graph $\widetilde{E}_7$ for character $\widetilde{u}_\bullet^0$.

(In the picture numbers mean order numbers of vertices.)



In this case $\widetilde{u}_\bullet^0 = (1, 1, 2, 3, 3, 0, 0, 0)$, $\widetilde{u}_\circ^0 = (0, 0, 0, 0, 0, 1, 1, 2)$.

Let us carry out an induction for $l = 2m$.

The base of induction is evident.

Let $c_{2m}(\widetilde{u}_\bullet^0) = (1, 1, 2, 3, 3, \rho(2m), \rho(2m), 2\rho(2m))$. Hence $c_{2m+2}(\widetilde{u}_\bullet^0) = (3 - \rho(2m), 3 - \rho(2m), 6 - 2\rho(2m), 9 - 3\rho(2m), 9 - 3\rho(2m), 4 - \rho(2m), 4 - \rho(2m), 8 - 2\rho(2m))$. Dividing all coordinates by $3 - \rho(2m)$ we will obtain a vector $(1, 1, 2, 3, 3, \rho(2m+2), \rho(2m+2), 2\rho(2m+2))$.

Now $\widetilde{u}_\bullet^{2m+1} = \overset{\circ}{c}(\widetilde{u}_\bullet^{2m}) = (1, 1, 2, 3, 3, (4 - \rho(2m)), (4 - \rho(2m)), (8 - 2\rho(2m)))$.

Object $(\pi, f) \in \operatorname{Rep}(G, \coprod')$ is *singular* if π is singular. Simplest object in the category $\operatorname{Rep}(G, \coprod')$ is a pair $(\Pi_g, \bar{f})$ $(d(\Pi_g) = \bar{g}$, $g \in G_v$, then $\bar{f}(g) = 0$ and $\bar{f}(x) > 0$ when $x \in M_g)$.

Proposition 1.4. *Each singular object of the category* Rep $(G, \coprod')$ *can be obtained in the form of* $\Phi_m(\Pi_g, \bar{f})$ *where* $(\Pi_g, \bar{f})$ *— simplest object (*$m \geq 0$ *when* $g \in \overset{\circ}{G}$ *and* $m \leq 0$ *when* $g \in \overset{\bullet}{G}$*). At that each faithful singular representation* G *corresponds (up to equivalence) to unique singular object* Rep $(G, \coprod')$.

Proof is carried out by the induction on $|t|$ where $t(d(\pi))$ is minimal in absolute value number, for which $c_t(d(\pi)) \notin V_G^+$.

Let $|t| = 1$. For definiteness $\overset{\circ}{c}\ (d(\pi)) \notin V_G^+$. Hence $\exists g \in G^\pi \bigcap \overset{\circ}{G}$, $f(g) = 0$ (in the contrary case we can apply to π the functor $\overset{\circ}{F}_{X,\delta}$ where $X = G^\pi$, δ is arbitrary, and will obtain $d(\overset{\circ}{F}_{X,\delta}\ (\pi)) = \overset{\circ}{c}\ (d(\pi)) \in V_G^+$). Then indecomposability of π and lemma 3.5 [7] imply $G^\pi = \{g\}$, i. e. $\pi = \Pi_g$, (π, f) is a simplest object.

Let the statement is true with $|t| < k$. Let (π, f) — singular object Rep $(G, \coprod')$, $c_k(d(\pi)) \notin V_G^+$; since π is not a simplest representation then f is positive on G^π. Hence either $c_{k-1}(d(\overset{\circ}{F}_{X,\delta}\ (\pi))) \notin V_G^+$ or $c_{k-1}(d(\overset{\bullet}{F}_{X,\delta}\ (\pi))) \notin V_G^+$ and we cane make use of the inductive presumption. □

Corollary 1.5. *Let* x *is a singular root of extended Dynkin graph* G*. Then* x *is a real root in* G*.*

Proposition 1.6. *Let* G *— extended Dynkin graph. In order to root* $x \in V_G$ *to be singular it is necessary and sufficiently that* $L_G(x) \neq 0$*.*

Proof.

Necessity. If x is singular then by the proposition 1.4 $x = c_t(\bar{g})$ where $\bar{g}$ is a simple root in G. Since $L(\bar{g}) \neq 0$ then, using the lemma 1.2, we will obtain the required.

Sufficiency. Consider along with $L_G(x)$ a linear form $L_G^+(x) = \sum\limits_{g_i \in \overset{\bullet}{G}} u_i x_i + \sum\limits_{g_j \in \overset{\circ}{G}} u_j x_j$. Reckon for definiteness $L_G(x) > 0$. Since $L_G(c_1(x)) = -L_G(x)$ (lemma 1.2) then $\sum\limits_{g_j \in \overset{\bullet}{G}} u_j x_j - \sum\limits_{g_j \in \overset{\bullet}{G}} u_j(\sigma(x_j))_j > 0$, and, consequently, $L_G^+(c_1(x)) < L_G^+(x)$. Analogously $L_G^+(c_2(x)) < L_G^+(c_1(x))$ etc. Thus, for the finite number of steps t we will obtain $L_G^+(x) < 0$, which means a singularity of x. □

Proposition 1.7. *If* G *— extended Dynkin graph then the presentation of each singular object of the category* Rep $(G, \coprod')$ *in the form of* $\Phi_m(\Pi_g, \bar{f})$ *is unique. (*$(\Pi_g, \bar{f})$ *is a simplest object,* $m \geq 0$ *when* $g \in \overset{\circ}{G}$ *and* $m \leq 0$ *when* $g \in \overset{\bullet}{G}$*).*

It follows from propositions 1.4 and 1.6. □

Proposition 1.8. *If* G *— extended Dynkin graph,* d *— singular root in* G *then there exists a unique standard representation* π *with the dimension* d*.*

Proof.

Proposition 1.4 imply $x = c_t(\bar{g})$. Let for definiteness $t \geq 0$. Hence $F_t(\Pi_g, \widetilde{u}^0_\bullet) = (\pi, \alpha\widetilde{u}^t_\bullet)$. Let $\pi' = \frac{1}{\sqrt{\alpha}}\pi$ (i. e. we multiply by $\frac{1}{\sqrt{\alpha}}$ each operator in the representation π). Then $\widetilde{u}^t_\bullet$ is a character of the representation π'.

Uniqueness π follows from the proposition 1.7. □

2 Generalization of function ρ.

Let $\widetilde{V}$ — set of infinite nonincreasing sequences of the integer nonnegative numbers $v = (v_1, v_2, \dots, v_i, \dots)$ such that $v_s = 0$ from the certain s. Define on $\widetilde{V}$ a partial order: $v \le w$, if $v_i \le w_i$ for all $i \in \mathbb{N}$.

Function ρ can be defined on set $\widetilde{V}$: for $v \in \widetilde{V}$ $\rho(v) = \sum_{i=1}^{m} \rho(v_i)$, $\rho(0) = 0$. Hence if $v \le w$ then $\rho(v) \le \rho(w)$.

Width $\omega(v)$ of a vector $v \in \widetilde{V}$ is a number of its nonzero components.

Let us introduce two following lists of vectors from $\widetilde{V}$:

$K = \{(1,1,1,1),\ (2,2,2)\ (3,3,1)\ (5,2,1)\}$;

$\widehat{K} = \{(1,1,1,1,1),\ (2,1,1,1)\ (3,2,2)\ (4,3,1)\ (6,2,1)\}$.

(Here and further, writing vectors from $\widetilde{V}$, we will write only their nonzero components.)

In the applications [10] one often encounter with conditions

$$\rho(v) = 4, \tag{1}$$

$$\rho(v) > 4. \tag{2}$$

Proposition 2.1. *[10]*

All solutions of the equation (1) are exhausted by the list K; all minimal solutions of the inequality (2) are exhausted by the list $\widehat{K}$.

Proposition 2.2. *If $\rho(w) > 4$ for $w \in \widetilde{V}$ then $\exists v \in \widetilde{V} \,|\, v < w$ and $\rho(v) = 4$.*

Proof follows from that fact that for any $\widehat{v} \in \widehat{K}$ $(\widehat{v} \le w)$ there exists such $v \in K$ that $v < \widehat{v}$. $\square$

Let us find out, what in propositions 2.1 and 2.2 is elementary and what is connected with the specificity of function ρ.

Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ is an arbitrary function. We will say that φ is *convex* if $\varphi(n+1) - \varphi(n) < \varphi(m+1) - \varphi(m)$ when $n > m$ and *normalized* if $\varphi(1) = 1$. (ρ possess these properties.) Let $\varphi(z_1, \dots, z_t) = \sum_{i=1}^{t} \varphi(z_i)$, $K(\varphi, n) = \{v \in \widetilde{V} \,|\, \varphi(v) = n\}$ and $N(\varphi, k)$ to consist of the minimal solutions of inequality $\varphi(v) > k$. For each $x \in \widetilde{V}$ correspond $\widehat{x} \in \widetilde{V}$ where $\widehat{x}_1 = x_1 + 1$ and $\widehat{x}_i = x_i$ when $i \ne 1$. Let $\langle s \rangle^r = \underbrace{s, \dots, s}_{r}$. Let $X \subset \widetilde{V}$, $|X| < \infty$, $\omega(X) = \max_{x \in X} \omega(x)$ and $\widehat{X} = \{\widehat{x} \,|\, x \in X\} \bigcup \{\langle 1 \rangle^{\omega(X)+1}\}$.

Following proposition is proved elementary.

Proposition 2.3. *Let $n \in \mathbb{N}$. If φ is increasing then $K(\varphi, n)$ and $N(\varphi, n)$ are finite. If, besides, φ is convex and normalized then $\widehat{K}(\varphi, n) \subset N(\varphi, n)$, and at that each vector from $\widehat{K}(\varphi, n)$ larger than precisely one vector from $K(\varphi, n)$. If $\widehat{K}(\varphi, n) = N(\varphi, n)$ and $m < n$ then $\widehat{K}(\varphi, m) = N(\varphi, m)$.*

Increasing, convex and normalized function φ is said to be *n-separating* ($n \in \mathbb{N}$) if $N(\varphi, n) = \widehat{K}(\varphi, n)$. It is equal to the following: if $\varphi(w) > n$, $w \in \widetilde{V}$ then there exists $v < w$ such that $\varphi(v) = n$.

Thus, ρ is 4-separating.

Let $v \in \widetilde{V}$, $t \in \mathbb{N}$. Let $v^{\langle t\rangle} = (v^{\langle t\rangle})_i$ where $(v^{\langle t\rangle})_i = v_1$ when $i \leq t$, and $(v^{\langle t\rangle})_i = v_{i-t}$ when $i > t$. For $X \subset \widetilde{V}$ let $X^{\langle t\rangle} = \{x^{\langle t\rangle}\}$ when $x \in X$.

Let us show that when $n > 4$ ρ is not n-separating and for $n = 4+i$ we construct n-separating functions ρ_i, such that $K(\rho_i, 4+i) = K(\rho, 4)^{\langle i\rangle}$ which (see§ 3) also connected with locally-scalar representations of quivers.

Consider an inequality

$$\rho(v) \geq 5. \tag{3}$$

Distinguish several cases.

1. $\omega(v) > 5$. Minimal vector satisfying this condition is vector $w = (1, 1, 1, 1, 1, 1)$, $\rho(w) > 5$. Since when $v < w$ $\rho(v) \leq 5$ then $w \in N(\rho, 5)$.

2. $\omega(v) = 5$. Under this condition $\rho(v) \geq 5$ and $\rho(v) = 5$ only if $v = (1, 1, 1, 1, 1)$ (v — the lowest vector in this case), i. e. $(1, 1, 1, 1, 1) \in K(\rho, 5)$. Then, obviously, $(2, 1, 1, 1, 1) \in N(\rho, 5)$.

3. $\omega(v) = 4$. Let $v = (v_1, v_2, v_3.v_4)$. If $v_4 \geq 2$ then $\rho(v) \geq 4\rho(2) = 5\frac{1}{3} > 5$. Hence $(2, 2, 2, 2) \in N(\rho, 5)$. If $v_4 = 1$, $v_3 \geq 2$ then $\rho(v) \geq \rho(1) + 3\rho(2) = 5$, in this case we obtain $(2, 2, 2, 1) \in K(\rho, 5)$, $(3, 2, 2, 1) \in N(\rho, 5)$. If $v_3 = v_4 = 1$ then the inequality reduces to $\rho(v_1) + \rho(v_2) \geq 3$. When $v_2 = 2$ we obtain $(5, 2, 1, 1) \in K(\rho, 5)$ and $(6, 2, 1, 1) \in N(\rho, 5)$. When $v_2 \geq 3$ we obtain $(3, 3, 1, 1) \in K(\rho, 5)$ and $(4, 3, 1, 1) \in N(\rho, 5)$.

4. $\omega(v) = 3$. Let $v = (v_1, v_2, v_3)$. Using reasonings, analogous to previous, we obtain:

a) when $v_3 \geq 5$ $(5, 5, 5) \in K(\rho, 5)$ and $(6, 5, 5) \in N(\rho, 5)$;

b) when $v_3 = 4$ $(9, 4, 4) \in K(\rho, 5)$ and $\{(10, 4, 4), (7, 5, 4)\} \subset N(\rho, 5)$;

c) when $v_3 = 3$ $\{(19, 4, 3), (11, 5, 3)\} \subset K(\rho, 5)$ and $\{(20, 4, 3), (12, 5, 3), (9, 6, 3)\} \subset N(\rho, 5)$;

d) when $v_3 = 2$ $\{(41, 6, 2), (23, 7, 2), (17, 8, 2), (14, 9, 2), (11, 11, 2) \subset K(\rho, 5)\}$; $\{(42, 6, 2), (24, 7, 2), (18, 8, 2), (15, 9, 2), (12, 11, 2), (13, 10, 2) \subset N(\rho, 5)\}$;

e) when $v_3 = 1$ we have $\rho(v) < \rho(1) + 2\rho(\infty) = 5$.

5. The case $\omega(v) \leq 2$ does not give any solutions, because $\rho(v) < 2\rho(\infty) = 4 < 5$.

Thus, the following proposition takes place:

Proposition 2.4. $K(\rho, 5) = \{(1, 1, 1, 1, 1), (2, 2, 2, 1), (3, 3, 1, 1), (5, 2, 1, 1), (41, 6, 2), (23, 7, 2), (17, 8, 2), (14, 9, 2), (11, 11, 2), (19, 4, 3), (11, 5, 3), (7, 7, 3), (9, 4, 4), (5, 5, 5).\}$. $N(\rho, 5) = \widehat{K}(\rho, 5) \bigcup \{(9, 6, 3), (7, 5, 4), (2, 2, 2, 2), (13, 10, 2)\}$

Let us fix $t \in \mathbb{N}$ and define a recurrent sequence $\{u_i\}$ in the following way:

$$u_0 = 0, u_1 = 1, u_{i+2} = tu_{i+1} - u_i. \tag{4}$$

Then, let us define a function $\rho_{t-2}(n)$:

$$\rho_{t-2}(n) = 1 + \frac{u_{n-1}}{u_n + 1} \text{ when } n \in \mathbb{N};\ \ \rho_{t-2}(0) = 0. \tag{5}$$

With $t = 2$ from (4) it is follows that $u_n = n$ and $\rho_0(n) = 1 + \frac{n-1}{n+1} = \rho(n)$, i. e. $\rho_0(n) = \rho(n)$. Let then $k = t - 2$, $k \in \mathbb{N}_0$.

Let us find a general formula for n-th member of sequence $\{u_n\}$. For that we solve a characteristic equation

$$\lambda^2 - (k + 2)\lambda + 1 = 0.$$

Its roots are $\lambda = \frac{k+2+\sqrt{k^2+4k}}{2}$, $\overline{\lambda} = \frac{k+2-\sqrt{k^2+4k}}{2}$.
Then formula of the n-th member (when $t \neq 2$) has the form

$$u_n = C_1\lambda^n + C_2\overline{\lambda}^n.$$

From the entry conditions we obtain $C_1 = \frac{1}{\sqrt{k^2+4k}}$, $C_2 = -\frac{1}{\sqrt{k^2+4k}}$.
Finally

$$u_n = \frac{1}{\sqrt{k^2+4k}}(\lambda^n - \overline{\lambda}^n).$$

After trivial transformations from the formula (5) we obtain

$$\rho_k(n) = 1 + \frac{\lambda^n - \lambda}{\lambda^{n+1} - 1}, \; k \neq 0. \tag{6}$$

Let us calculate

$$\rho_k(\infty) = \lim_{n\to\infty} \rho_k(n) = 1 + \overline{\lambda} = 1 + \frac{2}{k+2+\sqrt{k^2+4k}}. \tag{7}$$

Let us also show that the function $\rho_k(n)$ is increasing on $\mathbb{N}$. Using formula (6), the inequality $\rho_k(n) < \rho_k(n+1)$ reduces to the equivalent one $\lambda^3 + 1 > \lambda^2 + \lambda$ which is true due to $\lambda > 1$.

Define with $k > 0$ a function $\rho_k(n)$ on $\widetilde{V}$ analogously to the function ρ: for $v \in \widetilde{V}$ let $\rho_k(v) = \rho_k(v_1, \ldots, v_s) = \sum\limits_{i=1}^{s} \rho_k(v_i)$. Since ρ_k is increasing on $\mathbb{N}_0$ it is increasing also on $\widetilde{V}$.

Consider an inequality

$$\rho_k(v) \geq k + 4 \tag{8}$$

Let us show that the results will be analogous to results obtained for function ρ.

Lemma 2.5. *For any $k, n \in \mathbb{N}_0$ the following inequality is correct:*

$$\rho_k(n) + k\rho_k(\infty) < (k+1)\rho_k(n+1) \tag{9}$$

Proof.

When $k = 0$ the inequality reduces to the evident one $\rho(n) < \rho(n+1)$. Let $k > 0$. Using formulas (6) and (7), after transformations we obtain that the inequality (9) is equivalent to inequality

$$(\lambda^2 - 1)(\lambda^{n+2} - (k+1)\lambda^{n+1} + k) > 0,$$

which is true since $\lambda = \frac{k+2+\sqrt{k^2+4k}}{2} > k + 1$ when $k > 0$. $\square$

Proposition 2.6. *Function ρ_k is $(k+4)$-separating, $K(\rho_k, 4+k) = K^{\langle k\rangle}$ $(K = K(\rho, 4))$.*

$K(\rho_k, 4+k) = \{(\langle 1\rangle^{k+4});\ (\langle 2\rangle^{k+3});\ (\langle 3\rangle^{k+2}, 1);\ (\langle 5\rangle^{k+1}, 2, 1)\}$;

Proof.

On the ground of formula (5) we will find sequentially $\rho_k(1) = 1$, $\rho_k(2) = 1 + \frac{1}{k+3}$, $\rho_k(3) = 1 + \frac{1}{k+2}$, $\rho_k(4) = 1 + \frac{k+3}{k^2+5k+5}$, $\rho_k(5) = 1 + \frac{k+2}{k^2+4k+3}$.

Consider inequality (8).

1. Let $\omega(v) > k+4$. Then $\rho_k(v) \geq \rho_k(\langle 1\rangle^{k+5})$. Minimal vector among vectors with this width is vector $w = (\langle 1\rangle^{k+5})$. It is clear that for any $v < w$ (8) is correct, therefore $w \in N(\rho_k, k+4)$.

2. $\omega(v) = k+4$. $\rho_k(v) \geq \rho_k(\langle 1\rangle^{k+4}) = k+4$, and at that the equality here will be only if $v = (\langle 1\rangle^{k+4})$. Therefore, $(\langle 1\rangle^{k+4}) \in K(\rho_k, k+4)$. Minimal vector in this case is $w = (2, \langle 1\rangle^{k+3})$. The same way as in the previous case, for any $v < w$ will be $\rho_k(v) \leq k+4$, hence $w \in N(\rho_k, k+4)$.

3. $\omega(v) = k+3$. Let $v = (v_1, \ldots, v_{k+3})$.

a) Assume $v_{k+3} \geq 2$. Hence $\rho_k(v) \geq (k+3)\rho_k(2) \geq k+4$. Then the equality will be only if $v = (\langle 2\rangle^{t+1}) \in K(\rho_k, k+4)$, minimal vector is vector $w = (3, \langle 2\rangle^{t}) \in N(\rho_k, k+4)$.

b) Let $v_{k+3} = 1$ and $v_{k+2} \geq 2$. If $v_{k+2} \geq 3$ then $\rho_k(v) \geq \rho_k(1) + (k+2)\rho_k(3) = k+4$. So: $(\langle 3\rangle^{k+2}, 1 \in K(\rho_k, k+4))$, $(4, \langle 3\rangle^{k+1}, 1 \in N(\rho_k, k+4))$. Now consider $v_{k+2} = 2$. The initial inequality reduces to the following: $\rho_k(v_1) + \ldots + \rho_k(v_{k+1}) + \frac{1}{k+3} \geq k+2$. The equality is correct here if all $v_j = 5$, $j = \overline{1, k+1}$. Let us show that for all other vectors $v = (v_1, \ldots, v_{k+1}, 2, 1)$ in the case $v_{k+1} < 5$ will be $\rho_k(v) < \rho_k(\langle 5\rangle^{k+1}, 2, 1)$. Indeed, by the lemma 2.5 $\rho_k(v) < \rho_k(1) + \rho_k(2) + \rho_k(4) + k\rho_k(\infty) < \rho_k(1) + \rho_k(2) + (k+1)\rho_k(5) = \rho_k(\langle 5\rangle^{k+1}, 2, 1)$. And if $v_{k+1} \geq 5$ then vector $(\langle 5\rangle^{k+1}, 2, 1)$ will be, obviously, minimal. Thus, in this case we have $(\langle 5\rangle^{k+1}, 2, 1) \in K(\rho_k, k+4)$ and $(6, \langle 5\rangle^{k}, 2, 1) \in N(\rho_k, k+4)$.

c) $v_{k+3} = v_{k+2} = 1$. $\rho_k(v) < \rho_k(\langle\infty\rangle^{k+1}, 1, 1) = 2 + (n-1)\rho_k(\infty)$. Hence by the formula (7) $\rho_k(v) < k+3+\frac{k+1}{\lambda} < k+4$.

4. $\omega(v) \leq k+2$. $\rho_k(v) < (k+2)\rho_k(\infty) \leq k+4$ since $\rho_k(\infty) = 1 + \frac{1}{\lambda} \leq 1 + \frac{2}{k+2}$. In this case there are no solutions.

We considered all possible cases and, accordingly, we obtained that

$$K(\rho_k, k+4) = \{(\langle 1\rangle^{k+4}), (\langle 2\rangle^{k+3}), (\langle 3\rangle^{k+2}, 1), (\langle 5\rangle^{k+1}, 2, 1)\};$$

$$N(\rho_k, k+4) = \{(\langle 1\rangle^{k+4}), (2, \langle 1\rangle^{k+3}), (3, \langle 2\rangle^{k+2}), (4, \langle 3\rangle^{k+1}, 1), (6, \langle 5\rangle^{k}, 2, 1)\}.$$

□

Let $\alpha \in \mathbb{R}^+$. Consider functions $\rho_\alpha(v)$, $v \in \widetilde{V}$ with this assumption, defining them by the same rules (4) and (5) ($\alpha + 2 = t$). Now let us fix $v = (v_1, \ldots, v_s)$, $v \in \widetilde{V}$ and consider an equation

$$\rho_\alpha(v) = \alpha + 4 \tag{10}$$

where α is unknown.

Proposition 2.7. *Equation (10) has no rational non-integer solutions.*

Proof.

Formula (4) by induction evidently implies that u_n has a form $u_n = t^{n-1} + a_{n-2}t^{n-1} + \ldots + a_0$, $a_i \in \mathbb{Z}$, $i \in \mathbb{N}_0$. Let $v = (v_1, \ldots, v_s)$. Hence the equation (10) (taking into account (5)) will have a form

$$\sum_{i=1}^{s}(1 + \frac{u_{v_i - 1}}{u_{v_i} + 1}) = t + 2$$

$$\sum_{i=1}^{s}(\frac{t^{n-2} + b_{n-3}^{(i)}t^{n-3} + \ldots + b_0^{(i)}}{t^{n-1} + a_{n-2}^{(i)}t^{n-1} + \ldots + a_0^{(i)} + 1}) = t + 2 - s \ \ (a_j^{(i)}, b_j^{(i)} \in \mathbb{Z}).$$

After reduction to the common denominator we obtain an equation $P(t) = 0$ where $P(t)$ is a polynomial with coefficients from $\mathbb{Z}$, which leading coefficient equals 1. Consequently, $P(t)$ over field $\mathbb{R}$ has either integer or irrational roots. $\square$

Remark 1. *Proposition 2.7 is not true if assume $v_i \in \mathbb{N} \bigcup \{\infty\}$. For instance,* $\rho_{0,5}(1, 1, 1, \infty) = 4, 5$.

So, when $\alpha \in \mathbb{Q}^+$, the equality $\rho_\alpha(v) = \alpha + 4$ is possible only with integer α (proposition 2.7); for any $\alpha \in \mathbb{N}_0$ there exists a list of vectors satisfying this equality (proposition 2.6).

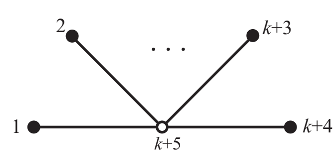
3 Standard representations of star graphs.

Graph G is said to be a *star graph* or a *graph of type* $(n_1, \dots, n_s)$ *($n_i \in N$, $i = \overline{1, s}$, $s > 1$)*, if G is a wood and G has exactly one vertex of multiplicity s (*nodal*), s vertices of multiplicity 1 (*extreme*), and the other vertices have multiplicity 2, at that the number of edges between nodal vertex and extreme ones equal $n_1, \dots, n_s$. The set of such graphs we denote S.

Let now $S_0 \subset S$ — the set of graph of the type $(n_1, \dots, n_s)$ such that $\rho_k(n_1, \dots, n_s) = k + 4$, $k \in \mathbb{N}_0$. The constitution of the set S_0 is determined by the proposition 2.6. Note that in S_0 extended Dynkin graphs $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ are contained (they correspond to the case $k = 0$).

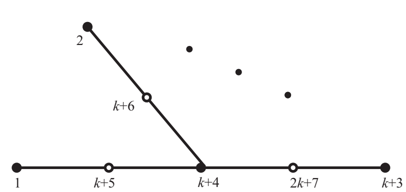
Let us consider locally-scalar representations of a graph from S_0. Further we will obtain a result analogous to the proposition 1.3. Below we specify for each family of graphs from S_0 even and odd standard characters. (In the pictures numbers mean the order number of vertices.)

1. $G = (\langle 1 \rangle^{k+4})$.



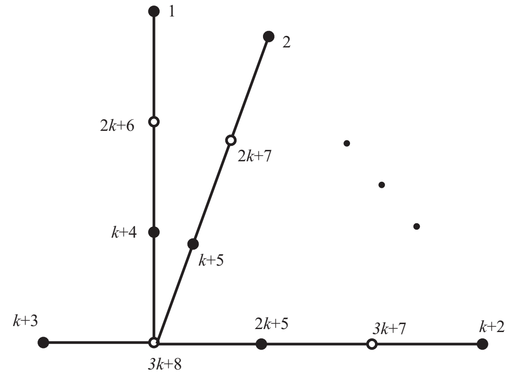
$\widetilde{u}^0_\bullet(k) = (\langle 1 \rangle^{k+4}, 0)$; $\widetilde{u}^0_\circ(k) = (\langle 0 \rangle^{k+4}, 1)$.

2. $G = (\langle 2 \rangle^{k+3})$.



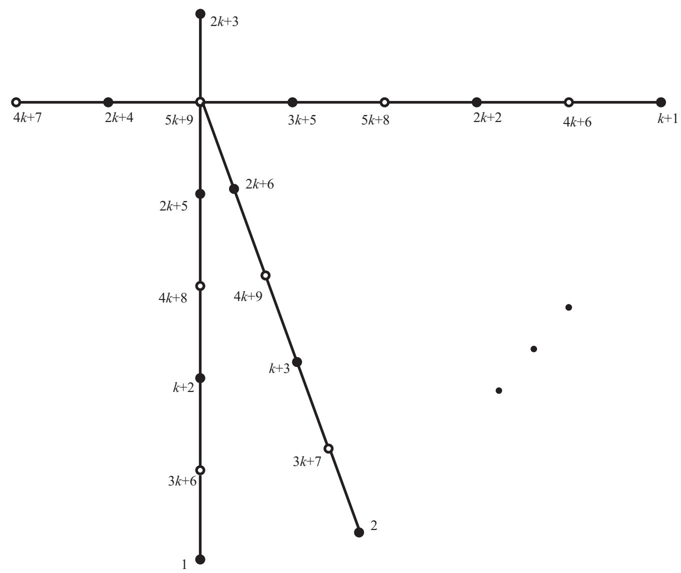
$\widetilde{u}^0_\bullet(k) = (\langle 1 \rangle^{k+3}, (k+3), \langle 0 \rangle^{k+3})$; $\widetilde{u}^0_\circ(k) = (\langle 0 \rangle^{k+4}, \langle 1 \rangle^{k+3})$.

3. $G = (\langle 3 \rangle^{k+2}, 1)$.



$\widetilde{u}^0_\bullet(k) = (\langle 1\rangle^{k+2}, k+2, \langle k+3\rangle^{k+2}, \langle 0\rangle^{k+3})$; $\widetilde{u}^0_\circ(k) = (\langle 0\rangle^{k+2}, \langle 1\rangle^{k+3}, k+2)$.

4. $G = (\langle 5\rangle^{k+1}, 2, 1)$.



$\widetilde{u}^0_\bullet(k) = (\langle 1\rangle^{k+1}, \langle k+3\rangle^{k+1}, (k^2+4k+3), (k^2+5k+4), \langle k^2+5k+5\rangle^{k+1}, \langle 0\rangle^{2k+2})$; $\widetilde{u}^0_\circ(k) = (\langle 0\rangle^{3k+2}, \langle 1\rangle^{k+1}, (k+1), \langle k+2\rangle^{k+1}, (k^2+4k+3))$.

Hence the following proposition, analogous to the proposition 1.3, takes place.

Proposition 3.1. *Let* $G \in S_0$, $\widetilde{u}^0_\bullet(t) = (w_1, \dots, w_p, 0, \dots, 0)$ *and* $\widetilde{u}^0_\circ(t) = (0, \dots, 0, w_{p+1}, \dots, w_n)$. *Then:*

$\widetilde{u}^{2m}_\bullet(t) = (w_1, \dots w_p, \rho_t(2m)w_{p+1}, \dots \rho_t(2m)w_n)$,

$\widetilde{u}^{2m+1}_\bullet(t) = (w_1, \dots w_p, (t+2-\rho_t(2m))w_{p+1}, \dots (t+2-\rho_t(2m))w_n)$,

$\widetilde{u}^{2m}_\circ(t) = (w_1, \dots w_p, (t+2-\rho_t(2m-1))w_{p+1}, \dots (t+2-\rho_t(2m-1))w_n)$,

$\widetilde{u}^{2m+1}_\circ(t) = (w_1, \dots w_p, \rho_t(2m+1)w_{p+1}, \dots \rho_t(2m+1)w_n)$,

$m \in \mathbb{N}_0$.

Proof is carried out analogous to the proof of the proposition 1.3 by the induction on $l = 2m$ separately for each family of graphs from S_0.

Formula for standard characters immediately implies

Proposition 3.2. *If* f *is a standard character then* $\overset{\circ}{f}$ *and* $\overset{\bullet}{f}$ *are standard character too. If* (π, f) *is a standard object of the category* $\mathrm{Rep}\,(G, \coprod')$ *then* $\overset{\circ}{F}(\pi, f)$ *is a standard object if* $\pi \neq \Pi_g$, *where* $g \in \overset{\circ}{G}$, *and* $\overset{\bullet}{F}(\pi, f)$ *is a standard object if* $\pi \neq \Pi_g$, *where* $g \in \overset{\bullet}{G}$.

Proposition 3.3. *If G is a star graph and $\rho_k(n_1, \ldots, n_s) = k+4$, d — singular root in G then there exists an unique singular f-representation π with dimension d, where f is a standard character.*

Proof of the existence is analogous to the proposition 1.8. The uniqueness folows from the proposition 3.2. □

References

[1] *P. Gabriel.* Unzerlegbare Darstellungen I – Manuscripta Math. 6 (1972), 71-103.

[2] *Bernstein I.N., Gelfand I.M., Ponomarev V.A.* Coxeter functors and Gabriel's theorem. – IMN, v. XXVIII, pt. 2, p. 19-33 (1973).

[3] *Nazarova L.A..* Representations of quivers of infinite type. – Izv. AN USSR. Ser. mat. 1973, **37**, p. 752–791.

[4] *Donovan P., Freislich M.R.* The representation theory of finite graphs and associated algebras. – Carleton Math. Lecture Notes, 1973, 5, p.1–119.

[5] *Kac V.G.* Some remarks on representations of quivers and infinite root systems. – Lect. Notes Math., 1980, **832**, p. 311-332.

[6] *P. Gabriel, A.V. Roiter* Representations of Finite-Dimensional Algebras. Berlin etc., Springer, 1997 – 171 p.

[7] *Kruglyak S.A., Roiter A.V.* Locally-scalar representations of graphs in the category of Hilbert spaces. – Present preprint.

[8] *Ostrovskii V., Samoilenko Yu.* Introduction to the Theory of Representations of Finitely Presented $*$-algebras. I. Representations by bounded operators. Rev. Math. & Math. Phys., vol.11, 1–261, Gordon and Breach, 1999.

[9] *Kruglyak S.A., Rabanovich V.I., Samoilenko Yu.S.* On sums of projectors – Functional analysis and its applications, v. 36, pt. 3, p. 20-35, 2002.

[10] *Nazarova L.A., Roiter A.V.* Norm of a relation, separating functions and representations of marked quivers. – Ukr. Mat. Jour., 2002, v. 54, N. 6.

[11] *Kac V.G.* Infinite root systems, representations of graphs and invariant theory, II. – I. Algebra, 1982, **78**, p. 141-162

[12] *Gelfand I.M., Ponomarev V.A.* Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space. – Coll. Math. Soc. J. Bolyai, 5. Hilbert Space Operators, Tihany (Hungary), 1970.